

Students' Mental Constructions and APOS Transitions in Matrix Learning: Implications for AI-Supported Learning Design

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Proceedings of the Kalijaga International Conference on Education 2026

Abstract. Students may perform matrix operations correctly while still demonstrating limited understanding of the mental constructions underlying those procedures. This study examined students' mental constructions in learning matrix multiplication and inverse matrix concepts using the Action, Process, Object, and Schema (APOS) framework, with particular attention to transitions between adjacent constructions and associated obstacles. A qualitative study was conducted with 23 Grade XI Social Sciences students at MA Nurul Ummah, Yogyakarta, Indonesia. Data were collected from students' written responses to two matrix activities and follow-up semi-structured interviews with three purposively selected students. The data were analyzed through APOS-guided coding using NVivo. Action was identified in 21 students (91%), Process in 17 students (74%), and Object in 4 students (17%). Four students (17.4%) demonstrated Action without corresponding evidence of Process, while 13 students (56.5%) demonstrated Process without corresponding evidence of Object. The Action-to-Process difficulty was characterized mainly by procedural reliance and limited explanations of mathematical meaning. The Process-to-Object difficulty was more prevalent and involved difficulties in coordinating procedures, representations, and the use of mathematical results in subsequent reasoning. These findings indicate that procedural performance does not necessarily reflect the development of more advanced APOS constructions. The findings also provide a basis for designing AI-supported learning as adaptive scaffolding through explanation, connection, generalization, and integration, although its effectiveness requires further empirical investigation.

Keywords: AI-supported learning; APOS theory; Learning obstacles; Mathematical thinking; Matrix learning

Abbreviations: APOS — Action, Process, Object, and Schema

INTRODUCTION

Matrices constitute an important topic in mathematics learning because they support various mathematical concepts and procedures, particularly matrix operations and systems of linear equations. However, students continue to experience difficulties in understanding matrix concepts, performing operations, and interpreting mathematical procedures. Studies have reported errors in matrix multiplication, determinants, and inverse matrices, indicating difficulties that extend beyond computation to conceptual understanding and appropriate application of procedures [1], [2], [3], [4], [5], [6], [7]. Thus, students' difficulties need to be examined beyond whether they obtain correct answers, particularly by considering how mathematical knowledge is understood and constructed.

Procedural performance does not necessarily reflect conceptual understanding. Research on matrix algebra has shown that students' mental constructions may remain concentrated at the Action and Process levels, with fewer students demonstrating Object construction and difficulties coordinating relationships among mathematical concepts [8]. Similar difficulties have been reported in matrix multiplication and systems of equations using inverse matrices, where students could execute procedures but experienced difficulties explaining or coordinating the underlying mathematical relationships [9], [10]. An APOS-based analysis of matrix learning likewise demonstrates the value of examining students' mental constructions rather than relying only on final answers [11].

APOS theory provides a framework for examining mathematical understanding through four mental structures (Action, Process, Object, and Schema), which develop through mechanisms such as interiorization and encapsulation [12], [13]. APOS has been applied to matrix algebra, matrix multiplication, matrix operations, and systems of equations using inverse matrices [8], [9], [10], [14]. These studies demonstrate its usefulness for examining how students construct mathematical knowledge in matrix-related contexts.

However, identifying students' mental constructions does not fully explain difficulties in developing subsequent constructions. Students may demonstrate an earlier APOS construction while experiencing

obstacles that hinder further development. Research has identified learning and cognitive obstacles in matrix learning and, from an APOS perspective, has shown that such difficulties may emerge at particular stages of mental construction [15], [16], [17], [18]. Studies of learning obstacles in other mathematical contexts further indicate that difficulties may involve conceptual, procedural, and representational aspects of mathematical activity [19], [20]. Therefore, analysis of mental constructions needs to be complemented by examination of the obstacles that may hinder subsequent constructions. Although previous studies have examined APOS constructions and learning obstacles, fewer have explicitly examined the presence and absence of subsequent APOS constructions at the student level and related these patterns to specific obstacles. Existing research has predominantly described students' constructions or difficulties within particular mathematical topics.

This study aims to examine students' mental constructions in learning matrix multiplication and inverse matrix concepts using the APOS framework. Specifically, the study identifies the mental constructions demonstrated across the APOS stages, examines patterns in the presence or absence of subsequent constructions across adjacent APOS stages, and analyzes the obstacles associated with these patterns. Based on these findings, the study further proposes implications for AI-supported learning design to provide potential scaffolding for students' development across APOS mental constructions.

METHODS

Study area

The study was conducted at MA Nurul Ummah, located in Kotagede, Special Region of Yogyakarta, Indonesia. The participants were 23 Grade XI Social Sciences students enrolled during the odd semester of the 2025/2026 academic year. The study focused on students' written responses to two matrix tasks presented in a student worksheet (LKPD), consisting of Activity 1 on matrix multiplication and Activity 2 on inverse matrices.

Procedures

Data collection

Data were collected from students' written responses to a student worksheet (LKPD) and follow-up semi-structured interviews. The LKPD consisted of two activities, namely Activity 1 on matrix multiplication and Activity 2 on inverse matrices. The LKPD was administered to 23 Grade XI Social Sciences students, who completed the tasks independently. The researcher did not provide instructional treatment during the completion of the LKPD, as the students' written responses served as the primary data for examining their mental constructions.

Participant selection and interview

After the students completed the LKPD, their written responses were examined for completeness. Three students with complete responses to both activities were purposively selected for semi-structured interviews. The interviews were conducted to clarify the reasoning underlying the students' written responses. The interview questions were developed based on each student's responses to the LKPD and focused on their understanding and reasoning related to matrix multiplication and inverse matrices.

Data analysis

The data were analyzed qualitatively using NVivo and guided by the APOS framework, focusing on students' Action, Process, and Object constructions [13], [21]. For coding purposes, Action was identified when students relied on explicit and sequential procedures without demonstrating an understanding of the meaning underlying the procedures; Process was identified when students demonstrated an internalized mental process, including the ability to explain relationships among procedural steps and manipulate or describe the process beyond explicit procedural execution; and Object was identified when students treated a mathematical process as a coherent entity that could be manipulated or subjected to new mathematical transformations without returning to the original procedure. The written responses of 23 students and interview transcripts from three selected students were coded into APOS categories and their corresponding child nodes. Matrix Coding Query was used to examine patterns of mental constructions across students and the two activities [22]. The analysis then examined the patterns of APOS constructions across the Action to Process and Process to Object stages to determine whether evidence of a subsequent mental construction was present in students' responses. Students who demonstrated an earlier construction but for whom evidence of the subsequent construction was not identified were examined as cases of an unsuccessful transition. In this study, an unsuccessful transition refers to the absence of evidence of the

subsequent APOS construction in the analyzed responses, rather than a direct observation of a temporal cognitive change. The relevant child nodes and associated conceptual, procedural, and representational and meaning-making obstacles were then examined. Interview data were used to clarify students' written responses and support source triangulation.

RESULTS AND DISCUSSION

Results

1. Students' Mental Constructions in Matrix Learning

The analysis of students' responses identified three APOS mental constructions Action, Process, and Object, across the matrix learning activities. The overall mental construction profiles of the 23 students are presented in Table 1.

Table 1. Students' Mental Construction Profiles Based on APOS Coding

Mental construction profile	Students	n
Action	S4, S15, S18, S19	4
Action–Process	S1, S3, S6, S7, S8, S9, S10, S14, S16, S17, S20, S22, S23	13
Action–Process–Object	S5, S12, S13, S21	4
No identified APOS coding	S2, S11	2
Total	23 students	23

As shown in Table 1, the most common mental construction profile was Action–Process, identified in 13 students. Four students showed only Action, while four students showed the complete Action–Process–Object pattern. Two students had no identifiable APOS coding in the analyzed data. The distribution of students showing each APOS component is presented in Table 2.

Table 2. Students Demonstrating Each APOS Mental Construction

APOS component	Students with coding	Percentage
Action	21/23	91%
Process	17/23	74%
Object	4/23	17%

Note. Percentages are based on the 23 students.

Action was identified in 21 of the 23 students (91%), while Process was identified in 17 students (74%). Object was identified in 4 students (17%). The coding references across the two learning activities are presented in Table 3.

Table 3. APOS Coding References across Learning Activities

APOS component	Activity 1: Matrix Multiplication	Activity 2: Inverse Matrix
Action	66	49
Process	70	30
Object	7	5

Note. Values represent the number of APOS coding references.

In Activity 1, the analysis produced 66 Action, 70 Process, and 7 Object coding references. In Activity 2, 49 Action, 30 Process, and 5 Object coding references were identified. The highest number of coding references in Activity 1 was associated with Process, whereas in Activity 2, Action had the highest number of coding references. Overall, the coding results indicate that the students' mental constructions were more frequently represented at the Action and Process levels, while Object appeared less frequently across the analyzed activities.

2. Evidence of Difficulties in the Transition of Mental Constructions

The analysis was further conducted to examine patterns across the APOS mental constructions. Rather than focusing only on the occurrence of Action, Process, and Object, the analysis examined whether evidence of a subsequent construction was identified following evidence of the preceding construction.

2.1. Transition from Action to Process

The Action-to-Process pattern was examined by cross-classifying the presence of Action and Process coding across the 23 students.

Table 4. Students' Transition from Action to Process

APOS transition	Number of students	Percentage
Action identified → Process identified	17	73.9%
Action identified → Process not identified	4	17.4%
Action not identified → Process identified	0	0%
Action not identified → Process not identified	2	8.7%
Total	23	100%

Note. Classification is based on the presence of Action and Process coding for each student.

The results show that 17 students (73.9%) demonstrated both Action and Process constructions. Meanwhile, 4 students (17.4%) demonstrated Action, but no corresponding evidence of Process was identified in their responses. No student showed Process without Action, while 2 students (8.7%) showed neither Action nor Process.

The four students in the Action identified → Process not identified category were S4, S15, S18, and S19. Their coded responses included procedural reliance and limited conceptual explanations.

Table 5. Evidence of Action Without Process

Student	Evidence identified
S4	Limited conceptual explanation; reliance on procedures
S15	Limited conceptual explanation; reliance on procedures
S18	Limited conceptual explanation; reliance on procedures
S19	Limited conceptual explanation; reliance on procedures

Thus, the four students showed evidence of Action, whereas no corresponding Process coding was identified in their data.

2.2. Transition from Process to Object

The Process-to-Object pattern was examined by cross-classifying the presence of Process and Object coding among the 23 students.

Table 6. Students' Transition from Process to Object

APOS transition	Number of students	Percentage
Process identified → Object identified	4	17.4%
Process identified → Object not identified	13	56.5%
Process not identified → Object identified	0	0%
Process not identified → Object not identified	6	26.1%
Total	23	100%

Note. Classification is based on the presence of Process and Object coding for each student.

The results show that 4 students (17.4%) demonstrated both Process and Object constructions, whereas 13 students (56.5%) demonstrated Process, but no corresponding evidence of Object was identified in their responses. No student demonstrated Object without Process, while 6 students (26.1%) showed neither Process nor Object. This overall classification was based on the student-level coding used for the 23-student analysis, whereas Table 7 presents the distribution of coded references by data source for the three students selected for interview.

The four students in the Process identified → Object identified category were S5, S12, S13, and S21, whereas 13 students demonstrated Process without corresponding Object evidence. To further examine how APOS constructions were evidenced across data sources, the responses of three students who demonstrated Action–Process–Object profiles (S12, S21, and S5) were compared across their LKPD responses and interviews. The coding results are presented by data source to show the distribution of references across the three APOS constructions.

Table 7. APOS Coding across Data Sources for Three Selected Students

Student	Data source	Action	Process	Object
S12	LKPD	12	2	0
S12	Interview	12	16	3
S21	LKPD	9	0	0
S21	Interview	10	9	2
S5	LKPD	9	4	0

Student	Data source	Action	Process	Object
S5	Interview	5	13	1

Note. Values represent the number of APOS coding references by data source.

The coding results differed across the two data sources. In the LKPD responses, no Object coding was identified for the three selected students. In contrast, Object coding was identified in the interview data for all three students, although the number of Object-coded references differed across students.

3. Conceptual and Procedural Obstacles Underlying the Patterns of Transition Difficulty

The analysis was conducted to identify the obstacles associated with students' difficulties in transitioning between APOS mental constructions. Four categories of obstacles were identified in the coded data: conceptual, procedural, representational and meaning-related, and prior-knowledge-related obstacles. The obstacle patterns were examined for the Action-to-Process and Process-to-Object transitions.

3.1. Obstacles Associated with the Action-to-Process Transition

Among the 23 students, four students (S4, S15, S18, and S19) showed evidence of Action, but no corresponding

Process evidence was identified in their responses. The obstacle coding for these students is presented in Table 8.

Table 8. Obstacles Identified among Students with Action but No Process

Obstacle	Number of students	Percentage
Conceptual obstacle	3	75.0%
Procedural obstacle	4	100.0%
Representational and meaning-related obstacle	4	100.0%
Prior-knowledge-related obstacle	0	0%

Note. A student may show more than one type of obstacle; therefore, percentages are not intended to sum to 100%.

3.2. Obstacles Associated with the Process-to-Object Transition

All four students showed procedural and representational and meaning-related obstacles, while three students also showed conceptual obstacles. No prior-knowledge-related obstacle was identified among these four students. The coded responses included procedural reliance and limited conceptual explanations.

The Process-to-Object transition showed a broader distribution of obstacles. Among the 23 students, 13 students showed evidence of Process, but no corresponding Object evidence was identified in their responses. The obstacle coding for these students is summarized in Table 9.

Table 9. Obstacles Identified among Students with Process but No Object

Obstacle	Number of students	Percentage
Conceptual obstacle	10	76.9%
Procedural obstacle	11	84.6%
Representational and meaning-related obstacle	9	69.2%
Prior-knowledge-related obstacle	0	0%

Note. A student may show more than one type of obstacle; therefore, percentages are not intended to sum to 100%.

Procedural obstacles were identified in 11 of the 13 students (84.6%), followed by conceptual obstacles in 10 students (76.9%) and representational and meaning-related obstacles in 9 students (69.2%). No prior-knowledge-related obstacle was identified in this group. The obstacle patterns occurred in different combinations across the students. For example, S1, S6, S7, S9, S10, S22, and S23 showed all three main obstacle categories, whereas other students showed fewer categories.

One representative LKPD response illustrates the difficulty in connecting a mathematical procedure with its conceptual meaning. In the response, the student applied the inverse-matrix formula and carried out the associated calculations, but the written explanation of the concept was limited.

Matriks A inverse adalah kebalikan dari matriks A itu sendiri.

$$\frac{1}{\det A} \begin{pmatrix} 1 & 3 \\ 3 & 4 \end{pmatrix} = \frac{1}{-2} \begin{pmatrix} 1-3 & \\ -2 & 4 \end{pmatrix}$$

$$\frac{1}{-2} \begin{pmatrix} 4 & -3 \\ -2 & 1 \end{pmatrix} = \begin{pmatrix} 2 & -\frac{3}{2} \\ 1 & -2 \end{pmatrix}$$

Figure 1. Example of a student's response showing the use of the inverse-matrix procedure with limited conceptual explanation.

3.3. Triangulation across LKPD and Interview Data

To further examine how obstacle evidence was expressed across data sources, the LKPD and interview data of three selected students, S12, S21, and S5, were compared. These students were selected from the Action–Process–Object profile to provide a cross-data-source comparison rather than to represent the unsuccessful transition groups. The results are presented in Table 10.

Table 10. Obstacle Coding across LKPD and Interview Data for Three Selected Students

Student	Data source	Conceptual	Procedural	Representational and meaning-related	Prior-knowledge-related
S12	LKPD	6	0	8	3
S12	Interview	19	2	8	5
S21	LKPD	7	2	4	0
S21	Interview	8	2	7	3
S5	LKPD	11	5	5	0
S5	Interview	10	0	6	4

Note. Values represent the number of coded references and may include multiple references from the same student.

The obstacle coding presented in Table 10 represents a separate triangulation analysis of the selected cases and is not included in the student-level frequencies reported in Tables 8 and 9. The coding showed that conceptual and representational and meaning-related obstacles were identified in both data sources for all three students, although the number of coded references differed. Procedural obstacle coding also varied across students and data sources. In contrast, prior-knowledge-related obstacles were identified in the interview data for all three students but were not identified in their LKPD responses. These findings were specific to the selected triangulation cases and were not part of the student-level frequencies reported in Tables 8 and 9.

The interview data further illustrated the difficulties identified in the written responses. For example, when S12 was asked to explain the procedure for determining the inverse matrix, he could describe the operation but was unable to explain why certain operations were performed:

Researcher: “Do you know why it is multiplied by negative or why it is switched?”

S12: “I don’t know.”

When asked about the relationship between matrix A, its inverse, and the equation $X = A^{-1}B$, he also stated:

Researcher: “Do you know the relationship between matrix A, the inverse of A, and the equation $X = A^{-1}B$?”

S12: “I don’t know.”

Together, these interview responses provide additional evidence of the difficulties reflected in the coding of the students’ written work.

Discussion

4. Why Were the Mental Transitions Difficult?

The findings indicate that difficulties in students’ APOS development were particularly visible in the transitions from Action to Process and from Process to Object. The four students who showed Action without Process evidence were characterized by procedural reliance and limited explanations of mathematical meaning. A larger group, 13 students (56.5%), showed Process without corresponding Object evidence, indicating difficulty in treating an established mathematical process as a coherent entity for

further manipulation. The following sections discuss these two patterns and the obstacles associated with them.

4.1. Why Was the Action-to-Process Transition Difficult?

The four students who showed Action without Process evidence relied mainly on explicit procedures and provided limited explanations of mathematical meaning. Their written responses showed procedural execution without sufficient

evidence of the relationships among the mathematical steps or their underlying meanings. Figure 1 presents S19's sequential matrix multiplication procedure.

$$A = \begin{Bmatrix} 3.000 & 2.000 \\ 2.500 & 1.500 \\ 4.000 & 3.000 \end{Bmatrix} \times B = \begin{Bmatrix} 3.000 & 2.000 \\ 2.500 & 1.500 \\ 4.000 & 3.000 \end{Bmatrix}$$

$$\left\{ \begin{array}{l} 3.000 \times 3.000 \\ 2.500 \times 3.000 \\ 2.000 \times 2.000 \\ 9.000 \times 5.000 \\ 2.000 \times 2.000 \\ 1.500 \times 2.000 \\ 3.000 \times 2.000 \end{array} \right\} \quad \left\{ \begin{array}{l} 3.000 \times 2.500 \\ 2.500 \times 2.500 \\ 1.5 \\ 7.000 \times 2.500 \\ 2.000 \times 1.500 \\ 1.500 \times 1.500 \\ 3.000 \times 1.500 \end{array} \right\} \quad \left\{ \begin{array}{l} 3.000 \times 4.000 \\ 2.500 \times 4.000 \\ 4.000 \times 4.000 \\ 2.000 \times 3.000 \\ 1.500 \times 3.000 \\ 3.000 \times 3.000 \end{array} \right\}$$

$$\left\{ \begin{array}{l} 6.000 + 7.500 + 12.000 \\ 7.500 + 5.000 + 10.000 \\ 12.000 + 10.000 + 16.000 \\ 4.000 + 3.000 + 6.000 \\ 3.000 + 3.000 + 9.500 \\ 6.000 + 4.500 + 9.000 \end{array} \right\}$$

Figure 2. S19's written work showing sequential matrix multiplication procedures.

Figure Figure 2 presents S18's reliance on a prescribed procedure without sufficient explanation of its mathematical meaning.

$$A = \begin{pmatrix} 3.000 & 2.000 \\ 2.500 & 1.500 \\ 4.000 & 3.000 \end{pmatrix} \times \begin{pmatrix} 3.000 & 2.000 \\ 2.500 & 1.500 \\ 4.000 & 3.000 \end{pmatrix}$$

$$A = \begin{pmatrix} 3.000 \times 3.000 & 3.000 \times 2.500 & 3.000 \times 4.000 \\ 2.500 \times 3.000 & 2.500 \times 2.500 & 2.500 \times 4.000 \\ 4.000 \times 3.000 & 4.000 \times 2.500 & 4.000 \times 4.000 \end{pmatrix}$$

$$= \begin{pmatrix} 9.000 & 7.500 & 12.000 \\ 7.500 & 6.250 & 10.000 \\ 12.000 & 10.000 & 16.000 \end{pmatrix}$$

Figure 3. S18's written response showing reliance on a procedure without sufficient conceptual explanation.

Taken together, these responses indicate that the students could carry out explicit mathematical actions, but their written work did not provide sufficient evidence that the actions had been coordinated into a coherent process. From an APOS perspective, this pattern suggests difficulty in interiorizing Actions into Processes. This pattern is consistent with previous APOS studies showing that students may demonstrate procedural activity while experiencing difficulties in coordinating mathematical relationships and developing deeper conceptual understanding [8], [9], [14]. The present study extends these findings by locating the difficulty specifically at the Action-to-Process transition, where procedural performance was evident but sufficient evidence of an internalized mathematical process was not.

4.2. Why Was the Process-to-Object Transition Difficult?

The Process-to-Object transition was more limited than the Action-to-Process transition. Although 17 students demonstrated Process coding, only 4 also demonstrated Object coding, while 13 showed Process without corresponding Object evidence. This pattern suggests that many students could perform or describe a mathematical process but did not yet demonstrate the ability to treat the resulting concept as a coherent entity for further manipulation. The responses of selected students who demonstrated Object coding provide additional insight into how Process and Object evidence emerged across data sources. S12, for example, followed the main procedural steps for determining the inverse matrix, including the use of the determinant and adjoint. However, the initial matrix representation in the written response was incorrect, leading to an incorrect determinant and incorrect elements in the adjoint matrix.

$$A^{-1} = \frac{1}{\det(A)} \times \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

$$= \frac{1}{-160.000} \times \begin{pmatrix} 100.000 & -110.000 \\ 110.000 & 160.000 \end{pmatrix}$$

$$= \begin{pmatrix} -\frac{100.000}{160.000} & \frac{110.000}{160.000} \\ \frac{110.000}{160.000} & -\frac{160.000}{160.000} \end{pmatrix}$$

Figure 4. S12's LKPD response showing the use of the inverse-matrix procedure

The interview data further showed that S12 could recall the operation but had difficulty explaining its underlying reason. When asked about the construction of the adjoint matrix, she described the procedural rule but could not explain why the operations were required:

Researcher: "Do you know how the adjoint is obtained?"

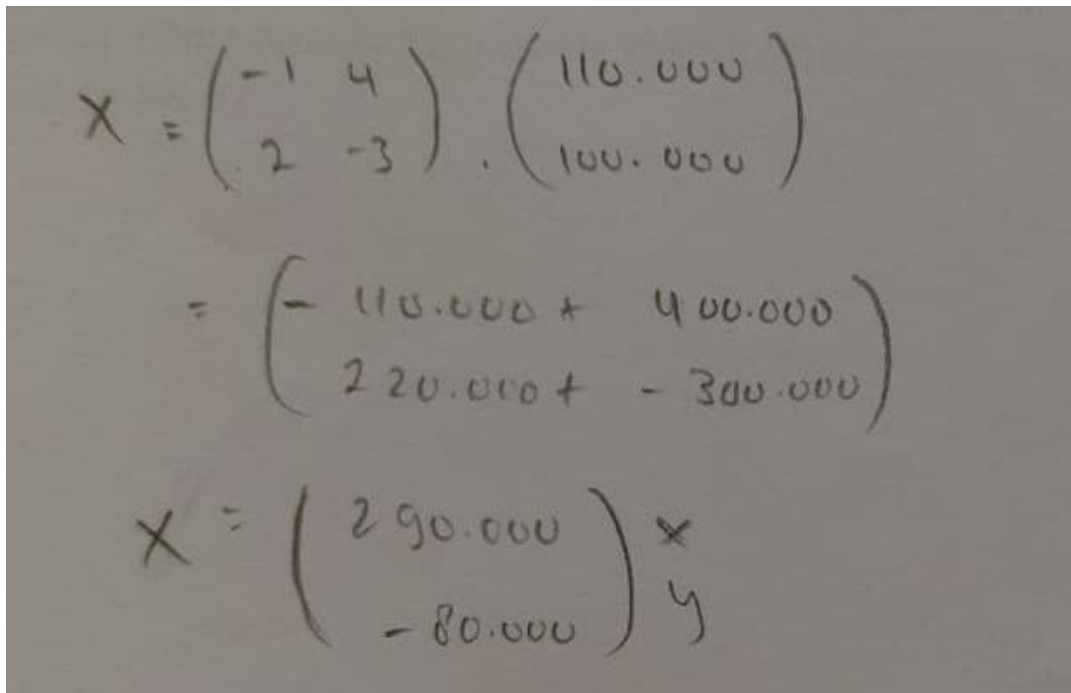
S12: "A and D are swapped, and B and C are multiplied by negative one."

Researcher: "But do you know why they are swapped or multiplied by negative one?"

S12: "I don't know."

This response suggests that procedural knowledge was present, but the mathematical relationships underlying the procedure were not fully articulated. The response therefore provides additional evidence of difficulty in coordinating the process and its mathematical meaning, which is relevant to the development of an Object construction. This difficulty was also evident when S12 was asked which part of the activity was most difficult. He identified the reflection activity and described the difficulty as finding the relationship. This response reinforces the interpretation that the difficulty extended beyond performing individual procedures to coordinating relationships among mathematical representations and processes.

A different aspect appeared in S5's work. The student had previously obtained the inverse matrix but, in the subsequent transformation, used the adjoint matrix rather than the inverse matrix obtained in the preceding step. Thus, although the inverse-matrix result had been produced, its role in the subsequent mathematical transformation was not used consistently.



$$X = \begin{pmatrix} -1 & 4 \\ 2 & -3 \end{pmatrix} \cdot \begin{pmatrix} 110.000 \\ 100.000 \end{pmatrix}$$

$$= \begin{pmatrix} -110.000 + 400.000 \\ 220.000 + -300.000 \end{pmatrix}$$

$$X = \begin{pmatrix} 290.000 \\ -80.000 \end{pmatrix} \begin{matrix} x \\ y \end{matrix}$$

Figure 5. S5's LKPD response showing the use of inverse-matrix components to determine X, with incomplete procedural manipulation

The interview data indicated that S5 recognized the need to use the inverse matrix but did not consistently maintain the relationship between the previously obtained result and its subsequent use. This suggests that, although Object evidence was identified for S5 in the interview data, the written response did not consistently demonstrate the use of the resulting mathematical entity in subsequent reasoning. These examples show that Process-to-Object development may not be expressed consistently across mathematical activities or data sources. More importantly, the student-level analysis showed that 13 students demonstrated Process without corresponding Object evidence. This pattern is consistent with APOS studies showing that Action and Process constructions may be present while Object construction remains limited, particularly when students have difficulty coordinating mathematical relationships or treating a process as an entity for further manipulation [23], [24]. The present study extends these findings by explicitly examining the presence and absence of subsequent constructions at the student level and relating the latter pattern to specific conceptual, procedural, and representational difficulties.

4.3. Interpreting the Failed Transitions through APOS

The two transition patterns indicate different difficulties in the construction of mathematical understanding. The Action-to-Process difficulty was characterized by reliance on explicit procedures and limited ability to explain the relationships and meanings underlying those procedures. In contrast, the Process-to-Object difficulty emerged when students could perform or describe mathematical processes but had difficulty treating the resulting concept as a coherent entity for subsequent manipulation.

From an APOS perspective, these patterns correspond to difficulties in interiorizing Actions into Processes and encapsulating Processes into Objects, respectively [13]. APOS research similarly shows that students may demonstrate Action and Process constructions while showing limited evidence of Object construction or experiencing difficulties because necessary mental structures have not been sufficiently developed [8], [9], [10], [23]. Recent research further indicates that difficulties may persist across different APOS stages, with errors occurring even at the Object stage and process-level difficulties remaining evident when students have not adequately constructed prerequisite understandings [24].

The present findings therefore suggest that procedural performance alone is insufficient evidence of higher mental construction. Difficulties became particularly visible when students were required to explain why a procedure was used, connect mathematical representations and meanings, or use a previously constructed result in a subsequent operation. Thus, examining both the APOS construction that is evidenced and the obstacles associated with the absence of a subsequent construction provides a more detailed account of students' mathematical development.

5. Implications for AI-Supported Learning

The findings of this study have implications for the design of AI-supported mathematics learning. The results indicate that students' difficulties were not limited to procedural performance, but were particularly evident in the transitions between Action and Process and between Process and Object. While Action coding was identified in 21 of 23 students (91%), Process coding was identified in 17 students (74%), and Object coding was identified in only 4 students (17%). In particular, 4 students showed Action without corresponding Process evidence, while 13 students demonstrated Process without corresponding Object evidence. These patterns suggest that AI support should not primarily function as an answer-generating tool, but should be designed to support the mental mechanisms required for progression across APOS constructions.

Recent research indicates that the educational value of generative AI depends on how its feedback and prompts are adapted to learners and instructional purposes. Adaptive LLM-generated feedback can support students' justifications, while LLM-based scaffolding can provide probing questions, targeted practice, feedback, and reflection [25], [26]. Systematic reviews of ChatGPT in education and mathematics education likewise highlight its potential for supporting teaching and learning while emphasizing the importance of how AI is pedagogically integrated [27], [28]. Moreover, prompt techniques can influence reflection, strategy use, representation, and other process-related qualities of AI-supported mathematical problem solving [29]. Based on these findings, the present study proposes an AI-supported learning sequence of Explain to Connect to Generalize and to Integrate, in which the function of AI progressively changes according to the mental construction that needs to be supported.

5.1. AI as an Explanation Elicitor for the Action-to-Process Transition

The Action to Process transition requires students to move beyond performing an explicit procedure toward understanding and mentally relating the actions involved. In the present study, some students were able to perform matrix procedures but had difficulty explaining why particular operations were used or how the steps were related. This pattern suggests a need for AI support that elicits explanations rather than simply supplying procedures or correct answers.

At this stage, AI support may be directed toward eliciting students' explanations of the reasoning underlying their procedures. Rather than providing a completed matrix multiplication or inverse-matrix procedure, the system could ask questions such as: Why is this row multiplied by this column? Why does this operation produce the element in this position? or What does this result represent in the context of the problem? Such prompts can encourage students to externalize the relationships underlying their actions and thereby support the internalization of actions into processes.

This proposed use is consistent with research showing that adaptive LLM-generated feedback can improve the quality of students' justifications and encourage more extensive engagement with feedback [25]. The importance of prompt design is also supported by [29], who found that prompt techniques influence process-related and pedagogical-contextual qualities of AI-generated mathematical solutions, particularly reflection and attention to common mistakes. Thus, AI support at this stage should be designed to prompt explanation and reflection, rather than replace the student's mathematical activity.

5.2. AI as a Connection Scaffold for Process Construction

The findings also indicate that some students were able to perform mathematical processes without consistently establishing relationships among procedures, representations, and the problem context. This suggests that support at the Process stage should focus on helping students connect mathematical actions and representations into a coherent mental process. For Process construction, AI support may be directed toward making relationships among procedures, representations, and contextual quantities explicit. For

example, the system could ask students to identify and explain relationships across representations: What does each row represent? What does each column represent? How is the resulting matrix related to the quantities in the problem? or why can this matrix multiplication be used to determine the shipping cost? Such prompts are intended to shift students from reproducing procedures toward explaining how the different components of the procedure work together.

This function is consistent with LLM-based scaffolding approaches that use probing questions, support for mathematical notation and symbols, targeted practice, immediate feedback, and reflection [26]. It is also consistent with evidence that effective generative-AI support depends on the alignment of prompts with learners' needs and the characteristics of the mathematical task [25], [29]. Thus, AI can be positioned not merely as a source of solutions but as a connection scaffold that prompts students to coordinate procedures, representations, and mathematical meanings.

5.3. AI as an Encapsulation Scaffold for the Process-to-Object Transition

The Process-to-Object transition represents the most critical implication of the present findings because only 4 of the 23 students (17.4%) demonstrated evidence of Object construction, whereas 13 students (56.5%) showed Process without corresponding Object evidence. The results suggest that many students could carry out or describe a mathematical process, but their responses did not provide clear evidence that the resulting concept had been treated as a mathematical entity that could be independently manipulated, interpreted, or used in subsequent reasoning. At this stage, the focus of AI support may shift from asking primarily how a procedure is performed to asking what the result of the process means and how it can be used. The AI could, for example, ask: What is the mathematical role of this inverse matrix?, What can you do with it after obtaining it?, How is the inverse matrix related to the original matrix? or Is the inverse matrix only the result of a calculation, or does it function as an object in another mathematical operation? Such prompts may encourage students to regard the process as a completed mathematical entity and to perform new actions on that entity.

This implication is particularly consistent with the APOS-grounded AI model proposed by Babaeisadr et al. [30], which explicitly links Process to the encapsulation of the process into an Object and includes analysis of students' understanding of the Object. Their study further reported that an integrated model combining AI, flipped learning, and

collaborative learning facilitated students' movement toward the Object and Schema stages, although the effect cannot be attributed to AI alone because the intervention combined several instructional components. This evidence supports the potential relevance of AI-supported APOS scaffolding while also indicating that such support should be embedded within a broader instructional design.

5.4. AI as a Schema-Building Facilitator

The present study did not identify Schema coding; therefore, movement toward Schema should be regarded as a pedagogical implication rather than an empirical finding. Nevertheless, the limited occurrence of Object construction suggests that further learning support is needed to help students integrate mathematical constructions into a more coherent and flexible structure. For prospective Schema development, AI support may be directed toward helping students connect previously constructed mathematical objects and processes across contexts. In the present topic, for example, an AI-supported learning environment could ask students to explain the relationship among the problem context, matrix representation, matrix multiplication, matrix equations, determinants, inverse matrices, and the interpretation of the resulting solution. The system could subsequently provide structurally varied problems and ask students to determine which concepts remain relevant, which relationships remain invariant, and how the same mathematical structure applies in a new context.

This proposed function is consistent with the APOS-oriented AI model of Babaeisadr et al. [30], which associates Schema with the integration and application of mathematical knowledge. More broadly, research on generative AI in mathematics education suggests that AI support can extend beyond answer generation toward reflection, strategic reasoning, and learning support when appropriately designed [27], [28], [29]. Thus, AI may potentially serve as a Schema-building facilitator, although this function remains to be empirically tested.

5.5. APOS-Oriented Direction for AI-Supported Learning

Based on the findings and the reviewed literature, AI support can be conceptualized as a progressively changing scaffold rather than a single instructional function.

Table 11. APOS-Oriented Functions of AI-Supported Scaffolding

APOS development	Need identified in this study	Proposed AI function	Illustrative AI support
Action → Process	Explain the reasons and relationships underlying	Explanation elicitor	Ask students to justify why a step is performed and explain

APOS development	Need identified in this study	Proposed AI function	Illustrative AI support
	procedures		its mathematical meaning
Process	Connect procedures, representations, and meanings	Connection scaffold	Ask students to relate rows, columns, matrix operations, results, and contextual quantities
Process → Object	Treat a completed process as a mathematical entity that can be manipulated	Encapsulation scaffold	Ask what the resulting matrix represents, what it can be used for, and how it relates to subsequent operations
Object → Schema	Integrate concepts and apply relationships flexibly across contexts	Schema-building facilitator	Ask students to connect concepts, identify structural relationships, compare representations, and solve structurally varied problems

Note. AI functions are proposed as design implications based on the study findings and literature.

Taken together, these proposed functions position AI as a progressively adaptive scaffold rather than a substitute for students' mathematical thinking. The sequence Explain to Connect to Generalize to Integrate represents increasing levels of support aligned with development from Action toward Process, Object, and, prospectively, Schema. The specific prompts should be determined by the mathematical task and student's demonstrated needs rather than applied as a fixed sequence. This orientation is consistent with research emphasizing adaptive feedback, learner-sensitive scaffolding, and the importance of prompt and task design in AI-supported learning [25], [26], [29].

Importantly, these AI functions are design implications derived from the present findings, not evidence that AI has been experimentally demonstrated to move students through all APOS stages. The APOS-oriented model proposed by Babaeisadr et al. [30], provides a relevant basis for connecting AI support with APOS development, but its combined instructional design does not isolate the effect of AI. Future research should therefore develop and test APOS-oriented AI scaffolds and examine whether these functions can facilitate the targeted mental transitions in actual mathematics learning environments. The effectiveness of such support will likely depend on the alignment among APOS construction, learner needs, mathematical task characteristics, and prompt design [29].

CONCLUSION

This study found that students' mental constructions in matrix learning were predominantly evidenced at the Action and Process stages, while Object construction was less frequently identified. The main difficulties occurred in the Action-to-Process and, more prominently, Process-to-Object transitions, which were associated with procedural, conceptual, and representational and meaning-related obstacles. These findings suggest that students may perform matrix procedures without fully internalizing their mathematical meaning or encapsulating the resulting processes as mathematical objects. Accordingly, AI-supported learning may provide potential scaffolding for these transitions

through explanation, connection, generalization, and integration, although this implication requires further empirical testing.

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