

Students' Strategies in Generalizing Geometric Patterns

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Abstract. This study aimed to classify the strategies used by junior high school students in generalizing mathematical patterns based on geometric patterns. Generalization was an important process in mathematics learning because it supported the formation of algebraic concepts and reasoning. The study was conducted using a descriptive qualitative approach. Based on the responses collected from 78 eighth-grade junior high school students in East Java. 40 were found to carry out generalization through the processes of relating, searching, and extending. The generalization processes demonstrated by these 40 students were subsequently categorized based on the strategies they employed.. Data were collected through written tests and in-depth interviews, then analyzed using the constant comparative (CCP) method. The results showed that there were three main strategies used by students in generalizing, namely: Un formula strategy, inductive strategies, and contextual strategies. These three strategies involved the stages of relating, searching, and extending according to the generalization framework. Although all students produced symbolic generalizations, the strategic approaches used had shown diversity in how the students thought and represented the patterns. This study recommended the importance of the role of teachers in facilitating various strategic approaches in pattern learning to improve students' mathematical generalization abilities.

Keywords: *Mathematical generalization, Inductive strategy, Geometric pattern, Symbolic generalization, Contextual strategy.*

INTRODUCTION

Various reasoning can be built through generalization [1]. Generalization is necessary in learning concepts in mathematics and it is also necessary in learning [2]-[3]. Furthermore, the curriculum standards also show the importance of generalization [4]. One of the important activities in mathematics learning is generalization [5], so generalization greatly supports mathematics learning [6]-[7]. The most authentic activity in mathematics learning is generalization [8]. Generalization is an important process in problem solving because the details of the solution are examined and investigated [9]. So it can be concluded that generalization plays a central role in the development of mathematical reasoning. Through generalization, learners are able to construct various forms of reasoning, making it a fundamental process in mathematics education. It is not only essential for understanding mathematical concepts but also widely recognized as a key component of effective learning.

Generalization activities are core activities of mathematics because they can build new knowledge [10]. Therefore, generalization is one of the important things in mathematics learning and learning activities. Students' mathematical thinking abilities are influenced by their generalization abilities [11]. Generalization means a process of obtaining general statements [6]. According to Krutetskii [6], generalization has two specific motives, namely to identify something common in specific cases and apply the general thing to the specific case, to find something general that is not yet known from the specific case. In addition, there are also those who define generalization as the process of finding a similarity in a case (specific thing) so that it can be applied generally [12].

Generalization refers to the process of identifying common features across individual examples or cases, enabling a conclusion that holds more broadly [12]-[13]-[14]. Peters described mathematics as a discipline concerned with the identification of patterns [15]. Patterns play a crucial role in mathematics and can indicate various approaches, including an understanding of counting numbers, number recognition, and the ability to relate numbers when identifying patterns. Number-pattern tasks involve problems centered on numerical sequences. In the context of problem solving in this study, recognizing number patterns entails examining the numbers within a given sequence and constructing a pattern, which enables the problem solver to generalize a solution applicable to all relevant situations [15]. In mathematics, there are two types of patterns, namely number patterns and geometric patterns [16]. In number patterns, learners are presented with a sequence of

numbers and are required to identify the pattern or relationship between consecutive terms in order to extend the sequence. Geometric patterns are number patterns that are represented diagrammatically, where the visual representation highlights the underlying structure of the number patterns [16].

Through patterns, complex generalizations can be made [17]. The patterns used can be numerical and geometric [18]-[19]. A sequence of numbers whose terms can be predicted or determined is called a pattern [20]. In geometric patterns, the next image can be obtained from the previous image with a procedure [21]. So, generalization in this study is a process consisting of relating, searching, and extending to find a general rule from a pattern. There are several types of mathematical patterns, recursive patterns consisting of repeating patterns and developing patterns [22]. Patterns are also classified into numerical patterns and geometric patterns [23]. Numerical patterns consist of numbers while geometric patterns consist of images.

According to Ellis, generalization is grouped into two, namely generalization action and reflective generalization [24]. Generalization action is a student's mental activity that is concluded based on the student's activities and conversations. In generalization action there are three categories: relating, searching and extending. Meanwhile, reflective generalization is a student's statement from the generalization [24]. In this relating action, students connect two or more problems or situations and can connect the similarities of two or more objects. In the searching action, students look for similar relationships, similar procedures, similar patterns or similar solutions or results. In the extending action, students expand the scope of application, eliminating specifics to develop broader cases, operating on an object to produce new cases or repeating existing patterns to produce new, broader cases [24]. The generalization process in this study uses this theory.

In the generalization process, there are several strategies that can be implemented, including explicit, whole-object, chunking, and recursive [25]. The explicit strategy is built by directly calculating each given output value based on certain input values. The whole-object strategy uses parts as units to build larger units. The chunking strategy builds a recursive pattern by constructing a sum based on known desired attribute values. The recursive strategy explains the relationship that occurs in situations between successive values of independent variables. In addition, there are also strategies for finding differences, namely: generalization strategies which focus on differences between provisions in number patterns [26]; Pattern finding strategies as generalization strategies which focused on the pattern formation of provisions from a number pattern [24]; quantity relationship strategies which include pattern generalization strategies and involve the relationship between input and output values [27]; trial and error strategies [28]; linear pattern strategies include a generalization strategy using linear pattern formulas [29]; visual grouping strategies and visual growth strategies [28]. The strategy which is often used by students is the recursive strategy, the strategy of finding the next n term using the previous n term [20]-[25]-[28].

Recent studies have primarily concentrated on the generalization strategies employed by students, particularly strategies for identifying differences, that is, generalization approaches that emphasize the differences between terms in number patterns [26]-[28]. Several previous studies investigating patterns from a sociocultural semiotic perspective have identified three types of generalizations of pre-symbolic and symbolic patterns made by eighth-grade students: factual generalization, contextual generalization, and symbolic generalization [30]. Radford develops a definition of algebraic generalization from an epistemological and semiotic perspective. Algebraic generalization must have three components: recognizing a fixed pattern in a sequence or structure, applying it implicitly to all previously unseen elements, and leveraging a general scheme or formula to express relationships between elements, for example using expressions like $n + 1$ or $2n + 1$. Factual generalization is based on operational observations, such as students observing concrete operations (e.g., adding fixed quantities), or generalizations that still take the form of actions, with no formal symbolic representation. In contextual generalization, students convey patterns in natural/informal language, such as "each image adds 2," naming variables or uncertainties explicitly but still within the context of words. While in symbolic generalization, students use algebraic symbols (e.g. $n + 1$ or $2n + 1$), a fully algebraic generalization is formed, where the mathematical relationship is transferred to a formula or formal scheme. Burgos' research on the levels of algebraic reasoning for elementary school students, based on an onto-semiotic approach [31].

The initial model developed in the onto-semiotic approach is combined with new subdivisions to produce sublevels for each level of algebraization. This serves as a tool for teachers and prospective teachers to understand in detail the thought processes and representations of students when learning algebraic concepts at the elementary school level. The expanded model includes several general levels which are then broken down into sublevels, including simple numerical generalizations (e.g., arithmetic operation patterns), semi-formal representations (using tables, diagrams, partial symbols), abstract generalizations with full symbols (using variables and formulas), and analytical and calculation processes (e.g., substitutions, basic algebraic manipulations). Each sublevel shows how students move from concrete intuition to symbolic expression and mathematical analysis.

Previous studies differ from this study in terms of the generalization theory used and the results obtained. Previous studies have not yet used the processes of relating, searching, and extending to describe students' strategies in generalizing geometric patterns. The generalization process in this study is described through generalizing actions (relating, searching, and extending) and reflective generalization (students' statements in their responses) [24]. The purpose of this study was to classify the strategies used by junior high school students in generalizing mathematical patterns through geometric patterns. The research question of this study is: What strategies are used by junior high school students in generalizing geometric patterns?

METHODS

Research Design

This research was conducted using a descriptive qualitative approach, appropriate for describing social processes or phenomena [32]. This approach used systematic procedures in analyzing data and guidelines for synthesizing descriptions of students' generalization processes. This aligned with the purpose of this study, which was to describe the classification of junior high school students' strategies in generalizing mathematical geometric patterns. Therefore, the data were qualitative, and the methodology chosen was also descriptive and qualitative.

Participants

The potential subjects in this study consisted of 25 students from SMP Negeri 3 Jombang, 27 students were from SMP Negeri 2 Sooko Mojokerto, and 26 students were from SMP Negeri 4 Kertosono. A total of 78 potential subjects were selected. These potential subjects were selected using a purposive sampling method. Based on the responses of 78 students, 40 students were identified as performing generalization through the processes of relating, searching, and extending. The generalization processes of these 40 students were then classified according to the strategies they used.

Instruments

The primary instrument was the researcher herself, who planned and collected data, interpreted data, analyzed data, concluded data, and reported the research results. Supporting instruments included questions on mathematical patterns and interview guidelines. Geometric patterns were chosen to be administered to research subjects because they these patterns could help junior high school students to develop fully functional strategies for discovering general formulas [33]. The instrument was validated by experts in mathematical thinking research and qualitative research in mathematics education. Furthermore, the mathematical pattern questions were also piloted on students who were not potential subjects. The question administered to potential subjects are as shown in Figure 1 below.

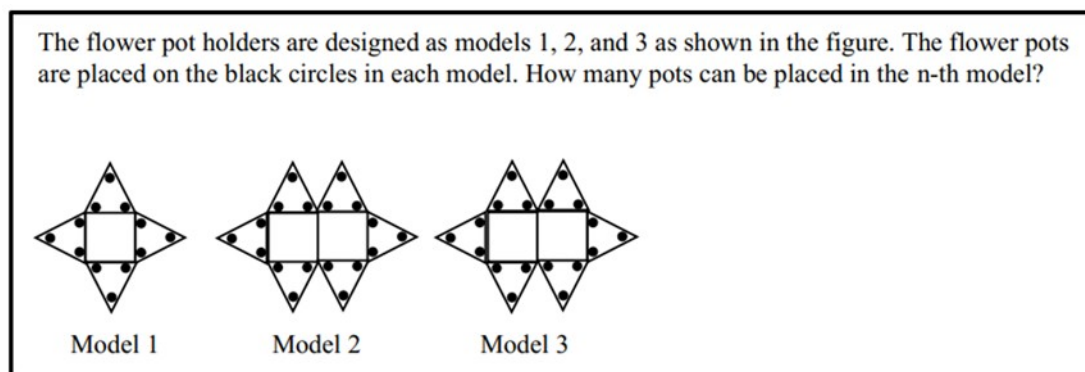


Figure 1. Mathematical Pattern Questions

Data Collection Procedures

The procedures for collecting research data were as follows: the researcher created research instruments, validated and conducted tests questions; 78 prospective subjects who were from three junior high schools had been given questions about patterns; checking students' answers to determine the research subjects or their answers were grouped according to the strategy used; grouping students' answers according to the strategy used; conducting interviews with students according to the groups formed, interviews were conducted based on students' answers to reveal more about the generalization process carried out by students; transcribing the results of interviews with research subjects; conducting data analysis.

Data analysis

The researchers used the constant comparative procedure (CCP) in data collection and analysis [32]-[34]. This inductive analysis method involved progressively developing categories of information by comparing one incident to another. CCP incidents proceeded from specific observations to general conclusions. This was

applied in this study to transform raw data from student answer sheets and in-depth interview transcripts into the generalization stages of relating, searching, and extending. Data were analyzed after 40 student responses were grouped according to the strategies used. One student from each group was selected for analysis of their answers and interview results. Figure 2 shows the CCP procedure used in this study.

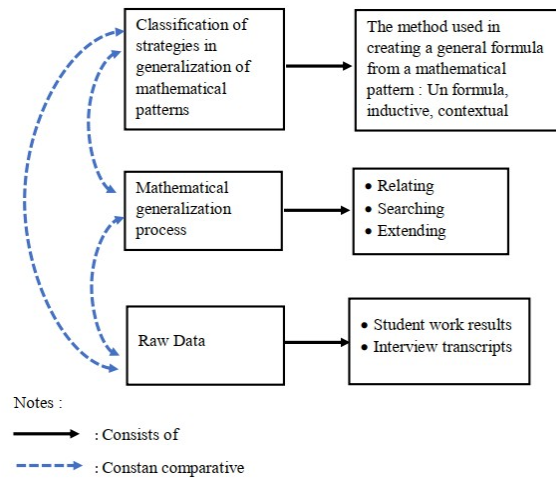


Figure 2. CCP Implementation in Data Analysis

Table 1. Indicators for Each Strategy

Generalization Process	Strategy		
	Un Formula	Inductive	Contextual
Relating	Calculating and relating the number of pots to the given model	Calculating and connecting the number of pots with the given model	Observing the elements of the model and relating each existing model
Searching	Identifying similarities in the differences in the number of pots across each model	Identifying a common procedure for calculating the number of pots that applies to all known models	Identifying pattern similarities based on the elements of the model
Extending	Determining the number of pots in the n th model using the formula Un	Applying the common procedure to determine the number of pots in the n th model	Applying the identified pattern to determine the number of pots in the n th model

RESULTS AND DISCUSSION

Results

The variability of students' strategies in generalizing mathematical patterns was demonstrated in the results of this study, which could be categorized according to observed general and consistent characteristics. Based on 40 student responses were grouped according to the strategies used, three strategies were classified. Table 2 showed the distribution of strategies used by students and their schools of origin.

Table 2. Distribution of Students' Strategies

No	School	Number of Students	Strategy Classification		
			Un formula	Inductive	Contextual
1.	SMP Negeri 3 Jombang	8	4	3	1
2.	SMP Negeri 4 Kertosono	12	7	3	2
3.	SMP Negeri 2 Soko Mojokerto	20	8	5	7
Total		40	19	11	10

The following table described the students' generalization process based on the strategy classifications used. Each classification was analyzed and described for one student, resulting in three students representing the research subjects.

Strategy I: using Un formula

Subject 1's answer was as shown in Figure 3. The generalization process based on written answers and interviews with Subject 1 was as follows:

The subject observed images of models 1, 2, and 3. The subject counted the number of pots in each model and wrote down the number of pots in each model. The subject then calculated the difference in the number of pots in models 1 and 2, and between models 2 and 3. The subject found the difference in the number of pots between models 1 and 2 was 6. The difference in the number of pots between models 2 and 3 was also 6. The subject also understood that the number of pots from model 1 to model 2 increased by 6, and likewise from model 2 to model 3 also increased by 6. Based on this, the subject used n th term formula (Un) to determine the

number of pots in model n. The nth term formula (U_n) was used to determine the n term of an arithmetic sequence. The formula was $U_n = a + (n - 1)b$, where a was the first term of an arithmetic sequence, n was the number of terms, and b was the common difference or difference of the arithmetic sequence. An arithmetic sequence was a sequence of numbers with a constant or fixed difference between two consecutive terms. The subject wrote the nth term formula (U_n), then plugged in each element and operated it until it was completed. Therefore, the number of pots in the n model was $6 + 6n$. The subject used nth term formula strategy (U_n).

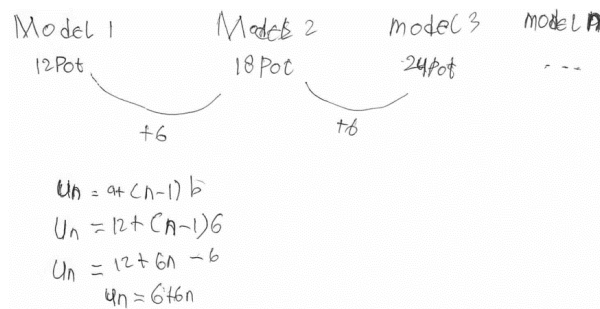


Figure 3. The Answer of Subject (Type 1)

Strategy II: using inductive strategy

Subject 2's answer was shown in Figure 4. The generalization process based on written answers and interviews with Subject 2 was as follows:

In the initial stage, Subject 2 observed the image in the problem and then counted the number of pots in each model. Subject 2 wrote down the number of pots in each model. Then, the subject sought a method for calculating the number of pots in each model. The subject experimented with methods that could be applied to each model. After experimenting, the subject discovered a procedure for calculating the number of pots that applied to each model. These methods were: for Model 1 $[(1 + 1) \times 6 = 12]$, Model 2 $[(2 + 1) \times 6 = 18]$, Model 3 $[(3 + 1) \times 6 = 24]$. The subject found a similar procedure for calculating the number of pots in Models 1, 2, and 3, and concluded that this procedure was also applied to Model n . Therefore, the number of pots in Model n was $(n + 1) \times 6$. The subject used an inductive strategy to determine the number of pots in model n .

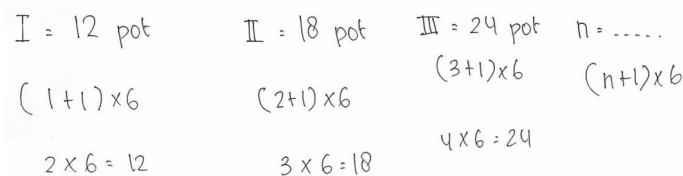


Figure 4. The Answer of Subject 2 (Type 2)

Strategy III: using contextual strategy

Subject 3's answer was as shown in Figure 5. The generalization process based on written answers and interviews with Subject 3 was as follows:

The subject observed the images in models 1, 2, and 3. The subject focused on the parts of the image. In model 1, there were 1 square and 4 triangles. The number of pots in model 1 was determined by multiplying the number of triangles by the number of pots in each triangle ($4 \times 3 = 12$). In model 2, there were 2 squares and 6 triangles. The number of pots in model 2 was determined by multiplying the number of triangles by the number of pots in each triangle ($6 \times 3 = 18$). In model 3, there were 3 squares and 8 triangles. The number of pots in model 3 was determined by multiplying the number of triangles by the number of pots in each triangle ($8 \times 3 = 24$). The subject understood that the number of squares from model 1 to the next always increased by 1, and likewise the number of triangles also increased by 2 from model 1 to the next successive model. So, the number of pots in model 1 to model 2 increased by 6 because there were two additional triangles from model 1 to model 2. Similarly, from model 2 to model 3, two additional triangles were also added. These additional triangles were the triangles above and below the square. The triangles on the right and left remained constant (as in Figure 6). Based on this, the subject concluded that the number of pots in model $n = n \times 2 \times 3 + 6$. n represented the number of squares in a particular model. The number 2 represented the number of triangles in each square (one upper triangle and one lower triangle). The number 3 represented the number of pots in each triangle. Meanwhile, the number 6 represented the number of pots in the right triangle (3 pots) and the left triangle (3 pots), which remained constant. The subject used contextually known elements of the model to determine the number of pots in model n .

Jika ada 1 persegi maka ada 2 segitiga (1 di atas dan 1 dibawah segitiga) dan 2 segitiga (1 kanan dan 1 kiri).
 1 segitiga ada 3 pot
 banyak pot n persegi = $n \times 2 \times 3 + 6$
 $= 2n \times 3 + 6$
 $= 6n + 6$

Translated Version

If there is 1 square, then there are 2 triangles (1 above and 1 below) and 2 triangles (1 on the right and 1 on the left).
 There are 3 pots in 1 triangle.
 Number of pots in n squares = $n \times 2 \times 3 + 6$
 $= 2n \times 3 + 6$
 $= 6n + 6$

Figure 5. Subject 3 Answer (Type 3)

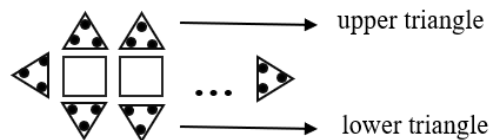


Figure 6. Illustration of adding squares and triangles

Discussion

Based on the results of the data analysis conducted in this study, it showed that out of 40 students who could generalize in applying the relation, searching, and extending processes [27]. Only 40 of these students consistently applied the relating, searching, and extending processes (could create general formulas from given patterns). This showed that junior high school students' generalization abilities still needed to be improved. In the relating action for subject 1 and subject 2, they connected the number of pots in a known model. However, it was different from subject 3, who connected each object in the model. In the searching action, subject 1 found the same difference for the number of pots in two consecutive patterns. Subject 2 performed a searching action by looking for the same rule for calculating the number of pots in each model. Meanwhile, subject 3 performed a searching action by looking for similar patterns in each known model. In the extending action, subject 1 expanded the range by using the n th term formula (U_n). Subject 2 in the extending action also expanded the range of similar procedures obtained in the searching action. Meanwhile, subject 3 performed an extending action by expanding the range based on the regularity of objects in the known model.

All subjects could create a general formula to determine the number of pots in the n model symbolically such as $6n + 6$ or $(n + 1) \times 6$ so that the subjects performed symbolic generalization [30]. Although the final results of all subjects were the same in being able to create a general formula, the strategies used were different. The strategy in classification III used contextual, this was different from the results of the study [30]. In previous studies, contextual generalizations were in the form of words not algebraically, but in this study the final result of a generalization was in the form of a general formula or algebraically/symbolically.

The generalization skills of junior high school students in Indonesia, especially for students in East Java needed to be improved. Students often had a lack of a clear understanding of the objectives of a problem, thus failing to utilize their ideas and experiences [35]. Students struggled to integrate knowledge and experience in problem-solving, which could hinder conceptual understanding [36]. According to NCTM, students had to represent an unknown quantity with letters or symbols [37]. In this research classification, all subjects used variables to determine the number of pots in the n -model.

CONCLUSION

This study showed that junior high school students' generalization abilities in constructing general rules from mathematical patterns could be classified into three strategies: n th term formula (U_n) as formal symbolic strategy, the inductive strategy, and the contextual strategy. These three strategies demonstrated that students' generalization process involved the stages of relating, searching, and extending. Although all three strategies

produced final generalizations in the form of symbolic expressions such as $6n+6$ or $(n+1) \times 6$, students' approaches and thinking in achieving these general rules varied. The n th term formula strategy (Un) tended to be based on numerical patterns and arithmetic sequences, the inductive strategy focused on repeated procedural observations, and the contextual strategy relied on interpretations of the visual structure and meaning of objects in the pattern.

These findings broadened our understanding of the variety of strategies students which were used to construct symbolic generalizations from geometric patterns, even when starting from a non-symbolic approach. They also demonstrated that symbolic generalizations were not solely generated through formal mathematical abstraction but could also be constructed through meaningful, contextual strategies connected to the visual representation of the pattern. However, only a small proportion of students were able to complete the complete generalization process, indicating that junior high school students' generalization abilities still needed to be improved. More systematic pedagogical interventions were needed to help students develop functional and symbolic thinking strategies, including providing diverse and exploratory patterns and training in representing mathematical relationships symbolically.

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